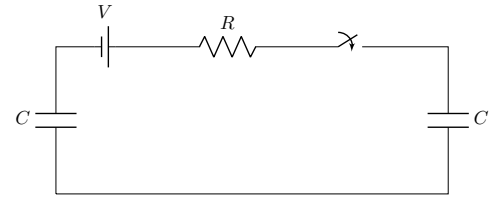


1. Two identical capacitors of capacitance C , an ideal battery of emf V and a resistor of resistance R are connected in series with a switch as shown in the figure. Before the switch is closed at time $t = 0$ both capacitors are uncharged.



- [5 Pts.] (a) Find the current in the circuit immediately after the switch is closed.

The two identical capacitors are in series, $\frac{1}{C_{\text{eq}}} = \frac{1}{C} + \frac{1}{C} \Rightarrow C_{\text{eq}} = \frac{C}{2}$.

Immediately after the switch is closed, both capacitors are uncharged and behave like short circuits.

$$I = \frac{V}{R}.$$

- [10 Pts.] (b) Find the current in the circuit as a function of time.

The circuit is equivalent to a resistor R in series with a capacitor of capacitance $C_{\text{eq}} = C/2$.

For a charging RC circuit, $V - \frac{dQ}{dt}R - \frac{Q}{C_{\text{eq}}} = 0 \Rightarrow Q(t) = C_{\text{eq}}V(1 - e^{-t/(RC_{\text{eq}})})$. Thus,

$$I(t) = \frac{V}{R}e^{-t/(RC_{\text{eq}})}.$$

Substituting $C_{\text{eq}} = C/2$ gives

$$I(t) = \frac{V}{R}e^{-2t/(RC)}.$$

- [5 Pts.] (c) What is the total energy lost on the resistor from $t = 0$ to $t \rightarrow \infty$?

The final charge on the equivalent capacitor is $Q_f = C_{\text{eq}}V = \frac{C}{2}V$.

So the final energy stored in the two-capacitor system is

$$U_f = \frac{1}{2}C_{\text{eq}}V^2 = \frac{1}{2}\left(\frac{C}{2}\right)V^2 = \frac{1}{4}CV^2.$$

The total work done by the battery is

$$W_{\text{bat}} = V\Delta Q = V\left(\frac{C}{2}V\right) = \frac{1}{2}CV^2.$$

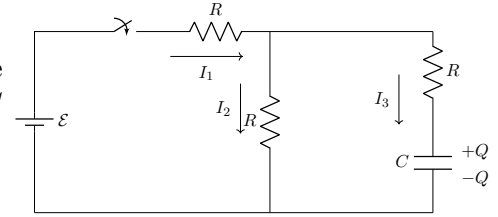
Hence the total energy dissipated in the resistor is

$$E_R = W_{\text{bat}} - U_f = \frac{1}{2}CV^2 - \frac{1}{4}CV^2 = \frac{1}{4}CV^2.$$

- [5 Pts.] (d) What is the total work done by the battery?

The total work done by the battery is $W_{\text{bat}} = V\Delta Q = \frac{1}{2}CV^2$

2. Consider the circuit shown in figure, where all resistors have the same resistance R . At $t = 0$, with the capacitor C uncharged, the switch is closed.



[10 Pts.]

- (a) At $t = 0$, find the values of three currents I_1 , I_2 , and I_3 .

At $t = 0$, the capacitor is uncharged, so it behaves like a short circuit. Therefore the right-hand branch is simply a resistor R , and the circuit consists of a resistor R in series with two equal resistors R in parallel. Hence the total resistance is

$$R_{\text{eq}} = R + \left(\frac{1}{R} + \frac{1}{R} \right)^{-1} = \frac{3R}{2}.$$

So the current through the top resistor is

$$I_1 = \frac{\mathcal{E}}{R_{\text{eq}}} = \frac{\mathcal{E}}{3R/2} = \frac{2\mathcal{E}}{3R}.$$

Since the two parallel branches have equal resistance, the current splits equally between them. Therefore,

$$I_2 = I_3 = \frac{I_1}{2} = \frac{\mathcal{E}}{3R}.$$

[10 Pts.]

- (b) At $t = \infty$, find the values of three currents I_1 , I_2 , and I_3 .

At $t = \infty$, the capacitor is fully charged and behaves like an open circuit. Therefore no current flows in the right-hand branch, so

$$I_3 = 0.$$

The remaining circuit consists of two resistors R in series, so the total resistance is

$R_{\text{eq}} = R + R = 2R$. Hence,

$$I_1 = I_2 = \frac{\mathcal{E}}{R_{\text{eq}}} = \frac{\mathcal{E}}{2R}.$$

[5 Pts.]

- (c) At $t = \infty$, what is the potential difference across the capacitor?

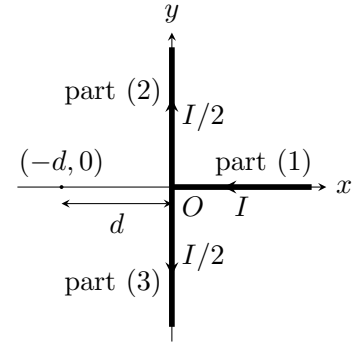
At $t = \infty$, no current flows through the right-hand resistor, so there is no voltage drop across it. Therefore the potential difference across the capacitor is equal to the potential difference between the top and bottom nodes, which is the same as the voltage across the middle resistor.

$$V_C = I_2 R = \left(\frac{\mathcal{E}}{2R} \right) R = \frac{\mathcal{E}}{2}.$$

3. An infinitely thin wire carries a steady current I along the positive x -axis from $x = +\infty$ to the origin. At the origin, the wire splits into two identical wires lying along the y -axis: one extends from $y = 0$ to $+\infty$, and the other from $y = 0$ to $-\infty$. Each of these wires carries a current $I/2$ away from the origin.

Label the segment along the x -axis as part (1), the segment along $+y$ as part (2), and the segment along $-y$ as part (3).

Hint: $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$, $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2+a^2}}$



[3 Pts.]

- (a) Find the magnetic field produced by part (1) at the point $(-d, 0)$.

Part (1) lies along the x -axis, and the point $(-d, 0)$ is on the same line. Hence, by the Biot-Savart law,

$$d\vec{B} \propto d\vec{\ell} \times \hat{r} = 0. \quad \Rightarrow \quad \vec{B}_1 = 0.$$

[10 Pts.]

- (b) Find the magnetic field produced by part (2) at the same point.

$d\vec{\ell} = dy \hat{y}$. And, $\vec{r} = -d \hat{x} - y \hat{y}$, $r = \sqrt{d^2 + y^2}$.

$$d\vec{\ell} \times \vec{r} = dy \hat{y} \times (-d \hat{x} - y \hat{y}) = d dy \hat{z}.$$

So by the Biot-Savart law

$$d\vec{B}_2 = \frac{\mu_0}{4\pi} \frac{(I/2) d\vec{\ell} \times \vec{r}}{r^3} = \frac{\mu_0 I}{8\pi} \frac{d dy}{(d^2 + y^2)^{3/2}} \hat{z}. \quad \Rightarrow \quad \vec{B}_2 = \frac{\mu_0 I d}{8\pi} \int_0^\infty \frac{dy}{(d^2 + y^2)^{3/2}} \hat{z}$$

Using integral above, $\int_0^\infty \frac{dy}{(d^2 + y^2)^{3/2}} = \frac{1}{d^2}$

$$\vec{B}_2 = \frac{\mu_0 I}{8\pi d} \hat{z}.$$

[6 Pts.]

- (c) Find the total magnetic field at the point $(-d, 0)$ due to all three parts of the wire.

By symmetry, part (3) produces a field at $(-d, 0)$ of the same magnitude but opposite direction: $\vec{B}_3 = -\frac{\mu_0 I}{8\pi d} \hat{z}$. Therefore,

$$\vec{B}_{\text{tot}} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 = 0 + \frac{\mu_0 I}{8\pi d} \hat{z} - \frac{\mu_0 I}{8\pi d} \hat{z} = 0.$$

[3 Pts.]

- (d) Determine the direction of the net magnetic force acting on part (2).

Only part (1) contributes to the force on part (2) because part (2) and part (3) are collinear. Hence, $d\vec{F} = \frac{I}{2} d\vec{\ell} \times \vec{B}_1$.

$$\hat{j} \times (-\hat{k}) = -\hat{i},$$

the force is directed toward the left.

[3 Pts.]

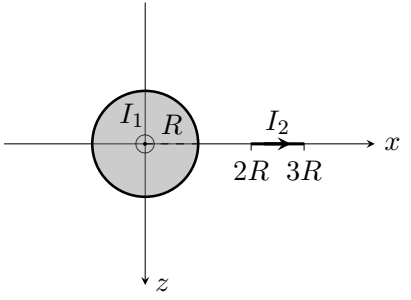
- (e) Find the total magnetic force acting on part (1).

At a point $(x, 0)$ on part (1), $\vec{B}_2(x, 0) = -\frac{\mu_0 I}{8\pi x} \hat{z}$, $\vec{B}_3(x, 0) = +\frac{\mu_0 I}{8\pi x} \hat{z}$.

These two fields are equal in magnitude and opposite in direction, so

$$\vec{B}_2(x, 0) + \vec{B}_3(x, 0) = 0, \quad \Rightarrow \quad \vec{F} = I d\vec{\ell} \times (\vec{B}_2 + \vec{B}_3) = 0.$$

4. An infinitely long cylindrical wire of radius R carries a steady current I_1 uniformly distributed over its cross section. The axis of the wire coincides with the y -axis, and the current flows in the $+y$ direction.



[5 Pts.]

- (a) Find the current density $\vec{J}$ inside the wire.

The magnitude of the current density is

$$J = \frac{I}{A} = \frac{I_1}{\pi R^2}.$$

Since the current flows in the $+\hat{y}$ direction, the current density vector is

$$\vec{J} = \frac{I_1}{\pi R^2} \hat{y}.$$

[10 Pts.]

- (b) Find the magnitude of the magnetic field inside the wire, at a distance $r < R$ from the axis.

Using Ampère's law:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{\text{enc}}.$$

$$B(r)(2\pi r) = \mu_0 I_{\text{enc}}.$$

The enclosed current is proportional to the enclosed area:

$$I_{\text{enc}} = I_1 \frac{\pi r^2}{\pi R^2} = I_1 \frac{r^2}{R^2}.$$

Therefore,

$$B(r) = \frac{\mu_0 I_1}{2\pi R^2} r.$$

[10 Pts.]

- (c) A second wire segment, assumed to be infinitely thin, lies along the positive x -axis from $x = 2R$ to $x = 3R$, and carries a current I_2 in the $+x$ direction. Determine the net magnetic force acting on this segment due to the magnetic field of the cylindrical wire. For points outside the cylindrical wire, the magnetic field due to the cylindrical wire is the same as that of a long straight wire carrying current I_1 along the $+\hat{y}$ direction. Thus, at a point on the positive x -axis at distance x from the axis,

$$\vec{B}(x) = -\frac{\mu_0 I_1}{2\pi x} \hat{k}.$$

A current element of the second wire is $d\vec{\ell} = dx \hat{i}$. Hence the magnetic force on this element is $d\vec{F} = I_2 d\vec{\ell} \times \vec{B}$. Using $\hat{i} \times (-\hat{k}) = \hat{j}$,

$$\vec{F} = \int_{2R}^{3R} \frac{\mu_0 I_1 I_2}{2\pi x} dx \hat{y} = \frac{\mu_0 I_1 I_2}{2\pi} \int_{2R}^{3R} \frac{dx}{x} \hat{y} = \frac{\mu_0 I_1 I_2}{2\pi} \ln\left(\frac{3}{2}\right) \hat{y}.$$